Exercise 1.3.2

Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_0-, t) = u(x_0+, t)$$

and no heat energy is lost at $x = x_0$ (i.e., the heat energy flowing out of one flows into the other). What mathematical equation represents the latter condition at $x = x_0$? Under what special condition is $\partial u/\partial x$ continuous at $x = x_0$?

Solution

If the heat energy flowing into $x = x_0$ is the same flowing out of it, then the energy balance gives

$$A(x_0-)\phi(x_0-, t) = A(x_0+)\phi(x_0+, t),$$

where ϕ is the heat flux, the thermal energy flowing per unit area, and A is the area. $A(x_0-)$ represents the area of the first rod and $A(x_0+)$ represents the area of the second one. Assuming the areas of the two rods are equal, this energy balance simplifies to

$$\phi(x_0-, t) = \phi(x_0+, t).$$

According to Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient.

$$\phi(x, t) = -K_0(x)\frac{\partial u}{\partial x}$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It has different values for different materials, so $K_0(x_0-)$ represents the thermal conductivity of the first rod and $K_0(x_0+)$ represents the thermal conductivity of the second one. The energy balance becomes

$$-K_0(x_0-)\frac{\partial u}{\partial x}(x_0-, t) = -K_0(x_0+)\frac{\partial u}{\partial x}(x_0+, t),$$

or

$$K_0(x_0-)\frac{\partial u}{\partial x}(x_0-, t) = K_0(x_0+)\frac{\partial u}{\partial x}(x_0+, t).$$

If the two rods happen to made of the same material, then $K_0(x_0-) = K_0(x_0+)$, and the energy balance implies that the temperature gradient will be continuous at $x = x_0$.

$$\frac{\partial u}{\partial x}(x_0-, t) = \frac{\partial u}{\partial x}(x_0+, t).$$